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ON THE SURFACE OF LOWEST DEGREE PASSING THROUGH A GIVEN CURVE IN SPACE.

By Tsuruichi Hayashi.

1. For the classification of curves in space, it is necessary to determine the lowest degree of the surfaces which contain a given curve.* For that purpose Salmon made use of the principle that a curve of degree m meets a surface of degree k in mk points, and many other investigators followed him. From this principle it follows that if we take on a curve of degree m as many points as are sufficient to determine a surface of degree k, and if the number of points so assumed be greater than mk, the surface described through the points must altogether contain the curve, for otherwise the principle would be violated.

Since the number of points through which a surface of degree k can be drawn is

$$\frac{k(k^2 + 6k + 11)}{6} = \varphi(k) \text{ say,}$$

when $\varphi(k) = mk + 1$, one and only one surface of degree k contains the given curve of degree m, and when $\varphi(k) > mk + 1$, two surfaces of degree k can be drawn, and therefore an infinite number.

If one or two surfaces of degree k contain the curve, two surfaces and therefore an infinite number of surfaces of degree higher than k contain the curve. For, if $\varphi(k) \geqq mk + 1$, then

$$\begin{aligned}\varphi(k + n) - m(k + n) - 1 & \\ & > \frac{(k + n)(\overline{k + n}^2 + 6\overline{k + n} + 11)}{6} - \frac{k^2 + 6k + 11}{6}(k + n) - 1 \\ & > \frac{n(k + n)}{6}(2k + n + 6) - 1 > 0,\end{aligned}$$

so that $\varphi(k + n) > m(k + n) - 1$.

2. Now let us proceed to discuss the equation $\varphi(k) = mk + 1$, which may be written

$$k^2 + 6k + 11 = 6m + \frac{6}{k}.$$

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* As my discussion is general, the word "general" should be used in speaking of curve and surface, thus "general curve" and "general surface." But for the sake of brevity I omit the word throughout.

Evidently k is 1, 2, 3 or 6. The last possibility does not give an integral value for m. Hence the only possibilities are

$$k = 1,\ m = 2; \qquad k = 2,\ m = 4; \qquad k = 3,\ m = 6.$$

The first case, that of a conic with a plane through it, is trivial. The other results may be stated thus:

Through the general quartic curve there passes a quadric surface, and through the general sextic a cubic surface.

We shall consider these two cases further.

In these two cases only, one and only one proper surface of degree k can be drawn containing the given curve in space. Two and only two classes of curves, quartic and sextic, can be divided into two families respectively, so that through a given curve belonging to the first family two and therefore more surfaces can be drawn, while through one belonging to the second family one and only one surface can be drawn. Remarks are necessary when the surfaces are not proper. For example, there are three kinds of quartic curves determined by a plane and a quartic surface, by two quadric surfaces, and by a quadric surface and a cubic surface; through any quartic curve of the first and second kinds, two and more quadric surfaces can be drawn, those for the first kind being improper quadrics, viz., quadrics resolved into two planes, one of which is fixed and the other is arbitrary; and through any quartic curve of the third kind, one and only one quadric surface can be drawn. There are two kinds of quintic curves determined by a quadric surface and a cubic surface, and by two cubic surfaces; through any quintic curve of either kind we can draw two and therefore more cubic surfaces. There are three kinds of sextic curves determined by a quadric surface and a cubic surface, by two cubic surfaces, and by a cubic surface and a quartic surface; through any sextic curve of the first and second kinds two and more cubic surfaces can be drawn, while through any one of the third kind one and only one cubic surface can be drawn. The other classes of curves, septic and of the higher degree, can not be divided into such two families. Through all of them we can draw two and therefore an infinite number of surfaces of a certain degree. That degree is given by the smallest value of k satisfying the inequality

$$\varphi(k) > mk + 1$$

for a given value of m.

3. Now let us find this smallest value of k. This inequality becomes

$$f(k) \equiv k^3 + 6k^2 - (6m - 11)k - 6 > 0.$$

If we put $k + 2 = \bar{k}$, the equation $f(k) = 0$ is reducible to the canonical form

$$\bar{k}^3 - (6m + 1)\bar{k} + 12(m - 1) = 0.$$

As the discriminant of this equation, namely

$$\left(\frac{12(m-1)}{2}\right)^2 - \left(\frac{6m+1}{3}\right)^3,$$

is negative for $m \geqq 2$, the three roots are real and distinct.

Applying the method of trigonometric solution of this irreducible cubic, we have that the three roots are

$$2\sqrt{\frac{6m+1}{3}}\cos\frac{\alpha}{3}, \qquad 2\sqrt{\frac{6m+1}{3}}\cos\frac{2\pi-\alpha}{3}, \qquad 2\sqrt{\frac{6m+1}{3}}\cos\frac{2\pi+\alpha}{3},$$

where α is the least positive value given by

$$\cos\alpha = -6(m-1)\sqrt{\frac{27}{(6m+1)^3}},$$

the radicals indicating the positive square roots.

Since $m \geqq 2$, $\pi/2 < \alpha < \pi$. Hence the first two of the above roots are positive and the third is negative. Moreover, the first is the larger of the positive roots. Therefore

Two and more surfaces of degree k can be drawn through the given curve of degree m, if k be selected not less than k_1, for which the relation

$$k_1 = 2\sqrt{\frac{6m+1}{3}}\cos\frac{\alpha}{3} - 2$$

holds, α being the least positive value satisfying

$$\cos\alpha = -6(m-1)\sqrt{\frac{27}{(6m+1)^3}}.$$

This gives the exact values of the greatest suffixes of the left A's in Halphen's tables in his memoir *Sur la classification des courbes gauches algébriques,* Journal de l'École Polytéchnique, vol. 52, 1882 [pp. 1–200], pp. 161–186.

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